

A variational representation and Prékopa's theorem for Wiener functionals

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Abstract

In 1998, Boué and Dupuis proved a variational representation for exponentials of bounded Wiener functionals. Since their proof involves arguments related to the weak convergence of probability measures, the boundedness of functionals seems inevitable. In this paper, we extend the representation to unbounded functionals under a mild assumption on their integrability. As an immediate application of the extension, we prove an analogue of Prékopa's theorem for Wiener functionals, which is then applied to formulate the Brascamp-Lieb inequality in the framework of Wiener spaces.

1 Introduction and main results

Let W be a standard d -dimensional Brownian motion. In [4] Boué and Dupuis showed the following representation for any bounded and measurable functional F that maps $C([0, 1]; \mathbb{R}^d)$ into \mathbb{R} :

$$\log \mathbb{E} [e^{F(W)}] = \sup_v \mathbb{E} \left[F \left(W + \int_0^\cdot v_s ds \right) - \frac{1}{2} \int_0^1 |v_s|^2 ds \right], \quad (1.1)$$

where the expectation \mathbb{E} is relative to W and the supremum is over all processes that are progressively measurable with respect to the augmentation of the natural filtration of W . In [4] the variational representation (1.1) was proven to be useful in deriving various large deviation asymptotics such as Laplace principles for small noise diffusions described by stochastic differential equations. These results have been extended by Budhiraja and Dupuis [7] to Hilbert space-valued Brownian motion, and later generalized by Zhang [29] to the framework of abstract Wiener spaces. In Boué-Dupuis [5], the representation (1.1) is also applied to risk-sensitive stochastic control problems.

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One of the purposes of this paper is to extend the representation (1.1) to any unbounded functional F that satisfies a certain integrability condition; the condition we impose is reasonably weak so that it allows F to diverge to $-\infty$ exponentially or faster at infinity (see Remark 1.1 below). We note that this is an essential extension; since the proof given in [4] relies on several results relevant to the weak convergence of probability measures (its Lemma 2.8 for example), the boundedness of the functional F seems inevitable. In this paper we use L^1 -convergence results such as Scheffé's lemma instead, and show that the boundedness of F is removable. Our reasoning is applicable to the setting of an abstract Wiener space as well. Recently in [27], Üstünel has extended the representation (1.1) to a class of unbounded functionals to characterize in terms of the relative entropy the invertibility of path transformations of Brownian motion W of the form $W + \int_0^\cdot v_s ds$. Our proof of the extension differs from his and the condition we draw on the functionals is considerably weaker than that imposed in [27]; see Remarks 1.1 and 2.1.

Prékopa's theorem states that, given a log-concave density function on a product of two finite-dimensional Euclidean spaces, say, $\mathbb{R}^m \times \mathbb{R}^n$, its n -dimensional marginal is also log-concave; this fact was originally proven by Prékopa [24] and then independently by Brascamp and Lieb [6] and Rinott [25]. As an application of the above-mentioned extension of (1.1), we prove an analogue of Prékopa's theorem for Wiener functionals. The derivation is straightforward once the extension of (1.1) is established. This analogue of Prékopa's theorem is then applied to extend the so-called Brascamp-Lieb moment inequality [6] to the framework of Wiener spaces; our argument bypasses any discretization steps and it allows us to formulate the inequality in a fairly general situation in which no specific regularities such as continuity are required for functionals involved in it. We also refer to Remark A.2 in the appendix for another motivation for the extension of (1.1). We note that by employing a finite-dimensionalization procedure, Prékopa's theorem is extended to the setting of an abstract Wiener space in Feyel and Üstünel [11]; our framework for the extension is wider than that of [11] in some respect, which enables us to recover original Prékopa's theorem from ours under the integrability condition associated with the extension of (1.1). See Remark 1.2.

We write \mathbb{W} for the space $C([0, 1]; \mathbb{R}^d)$ of all \mathbb{R}^d -valued continuous functions on $[0, 1]$ vanishing at the origin, equipped with the norm

$$|w|_{\mathbb{W}} := \sup_{0 \leq t \leq 1} |w(t)|, \quad w \in \mathbb{W}.$$

We denote by $\mathcal{B}(\mathbb{W})$ the associated Borel σ -field and by \mathbb{P} the Wiener measure on $(\mathbb{W}, \mathcal{B}(\mathbb{W}))$. In the sequel we denote by W the coordinate mapping process on \mathbb{W} :

$$W_t(w) := w(t), \quad 0 \leq t \leq 1, \quad w \in \mathbb{W}.$$

We set

$$\mathcal{F}_t := \sigma(W_s, 0 \leq s \leq t) \vee \mathcal{N}, \quad 0 \leq t \leq 1,$$

the filtration generated by W and augmented by the set \mathcal{N} of all \mathbb{P} -null events. We denote by \mathcal{A} the set of all \mathbb{R}^d -valued $\{\mathcal{F}_t\}$ -progressively measurable processes $v = \{v_t = (v_t^{(1)}, \dots, v_t^{(d)})\}_{0 \leq t \leq 1}$ satisfying

$$\int_0^1 \mathbb{E} [|v_t|^2] dt < \infty.$$

Here and in what follows, \mathbb{E} denotes the expectation with respect to \mathbb{P} and $|x|$ stands for the Euclidean norm of $x \in \mathbb{R}^d$.

Let $F : \mathbb{W} \rightarrow \mathbb{R}$ be measurable. We assume:

(A1) it holds that $\mathbb{E} [e^{F(W)}] < \infty$;

(A2) there exists $\delta > 0$ such that

$$\mathbb{E} [F_-(W)^{1+\delta}] < \infty,$$

where we set $F_-(w) := \max\{-F(w), 0\}$, $w \in \mathbb{W}$.

One of the main results of the paper is then stated as follows:

Theorem 1.1. *For any measurable function $F : \mathbb{W} \rightarrow \mathbb{R}$ satisfying (A1) and (A2), the following variational representation holds:*

$$\log \mathbb{E} [e^{F(W)}] = \sup_{v \in \mathcal{A}} \mathbb{E} \left[F \left(W + \int_0^\cdot v_s ds \right) - \frac{1}{2} \int_0^1 |v_s|^2 ds \right]. \quad (1.2)$$

We denote by \mathbb{H} the Cameron-Martin subspace of \mathbb{W} , namely \mathbb{H} consists of all elements $h = (h_1, \dots, h_d)$ in \mathbb{W} such that for each $i = 1, \dots, d$, the coordinate h_i is an absolutely continuous function whose derivative satisfies

$$\int_0^1 \left(\dot{h}_i(t) \right)^2 dt < \infty;$$

recall that \mathbb{H} is a Hilbert space with respect to the inner product

$$(h^1, h^2)_{\mathbb{H}} := \sum_{i=1}^d \int_0^1 \dot{h}_i^1(t) \dot{h}_i^2(t) dt, \quad h^1, h^2 \in \mathbb{H}.$$

For every $h \in \mathbb{H}$, we denote $|h|_{\mathbb{H}} = \sqrt{(h, h)_{\mathbb{H}}}$. The next theorem gives an analogue of Prékopa's theorem on the (classical) Wiener space $(\mathbb{W}, \mathbb{H}, \mathbb{P})$.

Theorem 1.2. *Let L be a real vector space and Λ a convex subset of L . We suppose $G : \mathbb{W} \times \Lambda \rightarrow \mathbb{R}$ to be such that:*

(B1) *for each $\lambda \in \Lambda$, the mapping $G(\cdot, \lambda) : \mathbb{W} \rightarrow \mathbb{R}$ is measurable and satisfies (A2);*

(B2) it holds that for any $w_1, w_2 \in \mathbb{W}$ with $w_1 - w_2 \in \mathbb{H}$, and for any $\lambda_1, \lambda_2 \in \Lambda$ and $\theta \in [0, 1]$,

$$\begin{aligned} & G(\theta w_1 + (1 - \theta)w_2, \theta \lambda_1 + (1 - \theta)\lambda_2) \\ & \geq \theta G(w_1, \lambda_1) + (1 - \theta)G(w_2, \lambda_2) - \frac{1}{2}\theta(1 - \theta) |w_1 - w_2|_{\mathbb{H}}^2. \end{aligned}$$

Then the mapping $\Lambda \ni \lambda \mapsto \log \mathbb{E} [e^{G(W, \lambda)}]$ is concave. Here we use the convention that $\log \infty = \infty$.

We give remarks on Theorems 1.1 and 1.2.

Remark 1.1. (1) Suppose that there exist constants $0 \leq C_1 < 1/2$, $0 < \alpha < 2$ and $C_2 \geq 0$ such that for \mathbb{P} -a.e. $w \in \mathbb{W}$,

$$\log(1 + F_-(w)) \leq C_2(1 + |w|_{\mathbb{W}}^\alpha) + C_1|w|_{\mathbb{W}}^2.$$

Then the assumption (A2) is fulfilled, which may be deduced from the fact (see, e.g., [16, Exercise 4.4.13]) that for all $a < 1/2$,

$$\mathbb{E} [\exp(a|W|_{\mathbb{W}}^2)] < \infty.$$

(2) Under the assumption (A1), the right-hand side of (1.2) is well-defined in the sense that for any $v \in \mathcal{A}$,

$$\mathbb{E} \left[F_+ \left(W + \int_0^\cdot v_s ds \right) \right] < \infty, \quad F_+ := \max \{F, 0\},$$

while $\mathbb{E} [F_- (W + \int_0^\cdot v_s ds)]$ may take the value ∞ for some $v \in \mathcal{A}$; see the proof of Proposition 2.1. As will also be seen below, the supremum over $v \in \mathcal{A}$ in the representation (1.2) can be replaced by that over all v 's in \mathcal{S} , a particular class of simple processes defined after Lemma 2.1. This replacement allows us to remove the assumption (A1) as to the well-definedness mentioned above because we have $\mathbb{E} [F_- (W + \int_0^\cdot v_s ds)] < \infty$ for all $v \in \mathcal{S}$; see Proposition 2.5.

(3) As shown in [4, Section 5], the representation (1.1) for any bounded F can be extended to any F which is only assumed to be bounded from below. This extension is a direct consequence of the monotone convergence theorem: For each positive real M , truncating F from above by M , we have from (1.1),

$$\log \mathbb{E} [e^{F_M(W)}] = \sup_{v \in \mathcal{A}} \mathbb{E} \left[F_M \left(W + \int_0^\cdot v_s ds \right) - \frac{1}{2} \int_0^1 |v_s|^2 ds \right],$$

where we set $F_M = \min \{F, M\}$. Then by the monotone convergence theorem, the left-hand side converges as $M \rightarrow \infty$ to the expression with F_M replaced by F , and so does

the right-hand side since

$$\begin{aligned}
& \sup_{M>0} \sup_{v \in \mathcal{A}} \mathbb{E} \left[F_M \left(W + \int_0^\cdot v_s ds \right) - \frac{1}{2} \int_0^1 |v_s|^2 ds \right] \\
&= \sup_{v \in \mathcal{A}} \sup_{M>0} \mathbb{E} \left[F_M \left(W + \int_0^\cdot v_s ds \right) - \frac{1}{2} \int_0^1 |v_s|^2 ds \right] \\
&= \sup_{v \in \mathcal{A}} \mathbb{E} \left[F \left(W + \int_0^\cdot v_s ds \right) - \frac{1}{2} \int_0^1 |v_s|^2 ds \right].
\end{aligned} \tag{1.3}$$

Therefore the essential part of Theorem 1.1 is the removal of the boundedness from below of F .

(4) In [27, Theorem 7], the representation (1.2) is proven to be valid under the condition that for some $p, q > 1$ with $p^{-1} + q^{-1} = 1$,

$$\mathbb{E} [|F(W)|^p] < \infty \quad \text{and} \quad \mathbb{E} [e^{qF(W)}] < \infty$$

while our assumption of Theorem 1.1 is equivalently rephrased as (A1) and $\mathbb{E} [|F(W)|^p] < \infty$ for some $p > 1$.

Remark 1.2. (1) Though the proof of Theorem 1.2 is easily done in its generality, the generalization to any real vector space L is not essential since the concavity is an expression on a line segment $\theta\lambda_1 + (1-\theta)\lambda_2$, $0 \leq \theta \leq 1$, for every fixed $\lambda_1, \lambda_2 \in \Lambda$.

(2) Let $g : \mathbb{R}^d \times \Lambda \rightarrow \mathbb{R}$ be such that

$$\text{the mapping } \mathbb{R}^d \times \Lambda \ni (x, \lambda) \mapsto g(x, \lambda) - |x|^2/2 \text{ is concave.} \tag{1.4}$$

Then the functional G defined by $G(w, \lambda) = g(w(1), \lambda)$, $(w, \lambda) \in \mathbb{W} \times \Lambda$, satisfies the condition (B2) of Theorem 1.2; indeed, letting $w_1, w_2, \lambda_1, \lambda_2$, and θ be as in (B2), we have from (1.4),

$$\begin{aligned}
& g(\theta w_1(1) + (1-\theta)w_2(1), \theta\lambda_1 + (1-\theta)\lambda_2) - \theta g(w_1(1), \lambda_1) - (1-\theta)g(w_2(1), \lambda_2) \\
& \geq -\frac{1}{2}\theta(1-\theta) |w_1(1) - w_2(1)|^2 \\
& \geq -\frac{1}{2}\theta(1-\theta) |w_1 - w_2|_{\mathbb{H}}^2,
\end{aligned}$$

where the second inequality follows from the fact that for any $h \in \mathbb{H}$,

$$|h(1)| = \left| \int_0^1 \dot{h}(t) dt \right| \leq |h|_{\mathbb{H}}.$$

Therefore Prékopa's theorem in finite dimension is recovered from Theorem 1.2 when the corresponding G defined above satisfies the integrability condition in (B1).

(3) The condition (B2) only concerns a pair of paths $w_1, w_2 \in \mathbb{W}$ whose difference is in \mathbb{H} , which we think reflects the fact that the structure of the Wiener space $(\mathbb{W}, \mathcal{B}(\mathbb{W}), \mathbb{P})$ is determined by its *skeleton* \mathbb{H} . Introduced by Feyel and Üstünel [11] is the notion of

\mathbb{H} -convexity, which, roughly speaking, is an almost sure convexity in the direction of \mathbb{H} . In Theorem 4.1 of [11], Prékopa's theorem is extended to the product of two abstract Wiener spaces, which we rephrase in our present setting as follows: if $G : \mathbb{W} \times \mathbb{W} \rightarrow \mathbb{R}$ is measurable and $\mathbb{H} \times \mathbb{H}$ -concave, namely

$$\begin{aligned} G(w_1 + \theta h^1 + (1 - \theta)k^1, w_2 + \theta h^2 + (1 - \theta)k^2) \\ \geq \theta G(w_1 + h^1, w_2 + h^2) + (1 - \theta)G(w_1 + k^1, w_2 + k^2) \end{aligned} \quad (1.5)$$

for $\mathbb{P} \times \mathbb{P}$ -a.e. $(w_1, w_2) \in \mathbb{W} \times \mathbb{W}$ for every $h^i, k^i \in \mathbb{H}$, $i = 1, 2$, and $\theta \in [0, 1]$, then the mapping

$$\mathbb{W} \ni w_2 \mapsto \log \int_{\mathbb{W}} e^{G(w_1, w_2)} \mathbb{P}(dw_1)$$

admits a version which is measurable and concave on \mathbb{W} . This assertion is proven by using finite-dimensional Prékopa's theorem and the Fourier expansion of elements in \mathbb{W} along a given complete orthogonal basis of \mathbb{H} . While the above condition is weaker than (B2) in the respect that it allows a negligible set on which the relation (1.5) fails, it does not allow the presence of the additional term $-(1/2)\theta(1 - \theta)|h^1 - k^1|_{\mathbb{H}}^2$ as in (B2); it seems difficult to draw such a term from a finite-dimensionalizing procedure as used in [11].

The rest of the paper is organized as follows: Section 2 is devoted to the proof of Theorem 1.1. The lower bound in the representation (1.2) is proven in Subsection 2.1; we prove the upper bound in Subsection 2.2 using the key Proposition 2.4 whose proof is given in Subsection 2.3; we also show in Subsection 2.3 a variant of Theorem 1.1 as Proposition 2.5, which is deduced from the proof of the theorem. In Section 3 we prove Theorem 1.2 and provide its application to the extension of the Brascamp-Lieb inequality to the framework of Wiener spaces. The proof of Theorem 1.2 is given in Subsection 3.1 by using Proposition 2.5; in Subsection 3.2 we formulate and prove the Brascamp-Lieb inequality on the Wiener space by applying Theorem 1.2. In the appendix, we discuss an extension of the Brascamp-Lieb inequality to the framework of nonconvex potentials in the case of one dimension.

For every $a, b \in \mathbb{R}$, we write $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$. For every $x, y \in \mathbb{R}^d$, we write $x \cdot y$ for the inner product of x and y in \mathbb{R}^d and denote $|x| = \sqrt{x \cdot x}$ as above. For every $1 \leq p \leq \infty$, we denote by $L^p(\mathbb{P})$ the set of all \mathbb{R} -valued random variables X defined on the probability space $(\mathbb{W}, \mathcal{B}(\mathbb{W}), \mathbb{P})$ such that

$$\{\|X\|_p\}^p := \mathbb{E}[|X|^p] < \infty \quad \text{for } p < \infty$$

and

$$\|X\|_\infty := \operatorname{ess\,sup}_{w \in \mathbb{W}} |X(w)| < \infty \quad \text{for } p = \infty.$$

Here and in what follows the notation $\operatorname{ess\,sup}_{w \in \mathbb{W}}$ stands for the essential supremum over $w \in \mathbb{W}$ with respect to \mathbb{P} . Other notation will be introduced as needed.

2 Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1.

For each $v \in \mathcal{A}$, we denote by T^v the path transform defined by

$$T_t^v(w) := w(t) + \int_0^t v_s(w) ds, \quad 0 \leq t \leq 1, \quad w \in \mathbb{W}.$$

We also set the process $\mathcal{E}^v = \{\mathcal{E}_t^v\}_{0 \leq t \leq 1}$ to be an $\{\mathcal{F}\}_t$ -local martingale defined by

$$\mathcal{E}_t^v := \exp \left(\int_0^t v_s \cdot dW_s - \frac{1}{2} \int_0^t |v_s|^2 ds \right), \quad 0 \leq t \leq 1.$$

In the case that \mathcal{E}^v is a true martingale, we define the probability measure \mathbb{P}^v on $(\mathbb{W}, \mathcal{B}(\mathbb{W}))$ by

$$\mathbb{P}^v(A) := \mathbb{E}[\mathbf{1}_A \mathcal{E}_1^v], \quad A \in \mathcal{B}(\mathbb{W}), \quad (2.1)$$

and denote by \mathbb{E}^v the expectation with respect to \mathbb{P}^v . By Girsanov's formula, the process $T^{-v}(W)$ is a standard Brownian motion under \mathbb{P}^v , which may be rephrased in the statement that the identity

$$\mathbb{E}^v [F(T^{-v}(W))] = \mathbb{E} [F(W)] \quad (2.2)$$

holds for any nonnegative measurable functional F on \mathbb{W} .

We say that an element v in \mathcal{A} is bounded if

$$\sup_{0 \leq t \leq 1} \|v_t\|_\infty < \infty.$$

The set of all bounded elements in \mathcal{A} will be denoted by \mathcal{A}_b . Well-known Novikov's condition implies that if $v \in \mathcal{A}_b$, then \mathcal{E}^v is a martingale. The following simple fact will also be referred to frequently:

Lemma 2.1. *Suppose that $v \in \mathcal{A}_b$. Then it holds that for any $p > 1$ and $0 \leq t \leq 1$,*

$$\mathbb{E}[(\mathcal{E}_t^v)^p] \leq \exp \left\{ \frac{1}{2} p(p-1) \sup_{0 \leq t \leq 1} \|v_t\|_\infty^2 \right\}.$$

Proof. By the definition of \mathcal{E}^v , we have

$$(\mathcal{E}_t^v)^p = \mathcal{E}_t^{pv} \exp \left\{ \frac{1}{2} p(p-1) \int_0^t |v_s|^2 ds \right\}.$$

Since the process \mathcal{E}^{pv} is also a martingale by the boundedness of v , we have $\mathbb{E}[\mathcal{E}_t^{pv}] = 1$ for all $0 \leq t \leq 1$, from which the claimed estimate follows readily. \square

We denote by \mathcal{S} the set of all \mathbb{R}^d -valued processes given in the form

$$v_t(w) = \xi_0 \mathbf{1}_{[t_0, t_1]}(t) + \sum_{k=1}^{m-1} \xi_k(w) \mathbf{1}_{(t_k, t_{k+1}]}(t), \quad 0 \leq t \leq 1, \quad w \in \mathbb{W}, \quad (2.3)$$

for some $m \in \mathbb{N}$, $0 = t_0 < t_1 < \dots < t_m = 1$, $\xi_0 \in \mathbb{R}^d$, and \mathbb{R}^d -valued bounded continuous functionals $\xi_k(w) = \xi_k(w(t), t \leq t_k)$, $w \in \mathbb{W}$, $k = 1, \dots, m-1$. We may deduce from [15, Lemma II.1.1] that \mathcal{S} is dense in \mathcal{A} with respect to the metric $\|\cdot\|_{\mathcal{A}}$ defined by

$$\|v\|_{\mathcal{A}}^2 := \mathbb{E} \left[\int_0^1 |v_s|^2 ds \right], \quad v \in \mathcal{A};$$

see also discussions in [16, Lemma 3.2.4, Problem 3.2.5] as to the density of \mathcal{S} in \mathcal{A}_b .

2.1 Proof of the lower bound

In this subsection we give a proof of the lower bound in (1.2), namely with the notation above, we prove

Proposition 2.1. *Assume that a measurable function $F : \mathbb{W} \rightarrow \mathbb{R}$ satisfies (A1). Then it holds that*

$$\log \mathbb{E} [e^{F(W)}] \geq \sup_{v \in \mathcal{A}} \left\{ \mathbb{E} [F(T^v(W))] - \frac{1}{2} \|v\|_{\mathcal{A}}^2 \right\}. \quad (2.4)$$

Remark 2.1. In [27, Theorem 6], the lower bound (2.4) is proven under the condition that $(1 + |F(W)|) e^{F(W)} \in L^1(\mathbb{P})$.

The proof of Proposition 2.1 is immediate if we are given the following lemma.

Lemma 2.2. *The lower bound (2.4) holds for any bounded and measurable F .*

Using this lemma, we prove Proposition 2.1.

Proof of Proposition 2.1. First we verify that under the assumption (A1),

$$\mathbb{E} [F_+(T^v(W))] < \infty \quad \text{for any } v \in \mathcal{A}, \quad (2.5)$$

where $F_+(w) := F(w) \vee 0$, $w \in \mathbb{W}$. Fix $v \in \mathcal{A}$ arbitrarily and set $F_{+,M} = F_+ \wedge M$ for each $M > 0$. Then by Lemma 2.2, we have in particular

$$\mathbb{E} [F_{+,M}(T^v(W))] \leq \log \mathbb{E} [e^{F_{+,M}(W)}] + \frac{1}{2} \|v\|_{\mathcal{A}}^2.$$

Letting $M \rightarrow \infty$, we apply the monotone convergence theorem to both sides to get

$$\begin{aligned} \mathbb{E} [F_+(T^v(W))] &\leq \log \mathbb{E} [e^{F_+(W)}] + \frac{1}{2} \|v\|_{\mathcal{A}}^2 \\ &\leq \log \mathbb{E} [1 + e^{F(W)}] + \frac{1}{2} \|v\|_{\mathcal{A}}^2, \end{aligned}$$

which is finite by (A1).

For every $M, N > 0$, we now define

$$F_N(w) := F(w) \vee (-N), \quad F_{N,M}(w) := F_N(w) \wedge M \quad \text{for } w \in \mathbb{W}.$$

Then by Lemma 2.2, the lower bound (2.4) holds for $F_{N,M}$. By letting $M \rightarrow \infty$, the monotone convergence theorem yields

$$\log \mathbb{E} [e^{F_N(W)}] \geq \sup_{v \in \mathcal{A}} \left\{ \mathbb{E} [F_N(T^v(W))] - \frac{1}{2} \|v\|_{\mathcal{A}}^2 \right\} \quad (2.6)$$

(cf. Remark 1.1 (3)). By the assumption (A1), the random variable $\sup_{N>0} e^{F_N(W)}$ is integrable and so is $\sup_{N>0} F_N(T^v(W))$ for any $v \in \mathcal{A}$ thanks to (2.5). Therefore as $N \rightarrow \infty$, we may use the monotone convergence theorem on both sides of (2.6) to obtain

$$\begin{aligned} \log \mathbb{E} [e^{F(W)}] &\geq \inf_{N>0} \sup_{v \in \mathcal{A}} \left\{ \mathbb{E} [F_N(T^v(W))] - \frac{1}{2} \|v\|_{\mathcal{A}}^2 \right\} \\ &\geq \sup_{v \in \mathcal{A}} \inf_{N>0} \left\{ \mathbb{E} [F_N(T^v(W))] - \frac{1}{2} \|v\|_{\mathcal{A}}^2 \right\} \\ &= \sup_{v \in \mathcal{A}} \left\{ \mathbb{E} [F(T^v(W))] - \frac{1}{2} \|v\|_{\mathcal{A}}^2 \right\}, \end{aligned}$$

which shows the proposition. \square

The statement of Lemma 2.2 is the same as what is proven in the first half of the proof of Theorem 3.1 in Boué-Dupuis [4]. For the self-containedness of the paper, we give a proof of the lemma, which slightly differs from and simplifies the original one.

We begin with the next two lemmas, assertions of which are taken respectively from pages 1648 and 1649 of [4].

Lemma 2.3. *Let $F : \mathbb{W} \rightarrow \mathbb{R}$ be bounded and measurable. It holds that for any $v \in \mathcal{A}_b$,*

$$\log \mathbb{E} [e^{F(W)}] \geq \mathbb{E}^v \left[F(W) - \frac{1}{2} \int_0^1 |v_s|^2 ds \right].$$

Proof. It is readily seen that

$$\begin{aligned} \log \mathbb{E} [e^{F(W)}] - \mathbb{E}^v [F(W) - \log \mathcal{E}_1^v] &= \mathbb{E}^v \left[\log \left(\frac{\mathbb{E} [e^{F(W)}] \mathcal{E}_1^v}{e^{F(W)}} \right) \right] \\ &\geq \mathbb{E} \left[\left(1 - \frac{e^{F(W)}}{\mathbb{E} [e^{F(W)}] \mathcal{E}_1^v} \right) \mathcal{E}_1^v \right] \\ &= 1 - 1 = 0. \end{aligned}$$

Here for the second line we used the inequality $\log x \geq 1 - 1/x$ for all $x > 0$, and the definition (2.1) of \mathbb{P}^v . The proof of the lemma ends by noting that

$$\begin{aligned}\mathbb{E}^v [\log \mathcal{E}_1^v] &= \mathbb{E}^v \left[\int_0^1 v_s \cdot dW_s - \int_0^1 |v_s|^2 ds \right] + \mathbb{E}^v \left[\frac{1}{2} \int_0^1 |v_s|^2 ds \right] \\ &= \mathbb{E}^v \left[\frac{1}{2} \int_0^1 |v_s|^2 ds \right]\end{aligned}\tag{2.7}$$

because of the fact that the process

$$\left(\int_0^t v_s \cdot dW_s - \int_0^t |v_s|^2 ds \right) \mathcal{E}_t^v, \quad 0 \leq t \leq 1,$$

is a martingale by Itô's formula, the boundedness of v and Lemma 2.1. \square

Lemma 2.4. *Let $F : \mathbb{W} \rightarrow \mathbb{R}$ be bounded and measurable. It holds that for any $v \in \mathcal{S}$,*

$$\log \mathbb{E} [e^{F(W)}] \geq \mathbb{E} [F(T^v(W))] - \frac{1}{2} \|v\|_{\mathcal{A}}^2.\tag{2.8}$$

Proof. Let $v \in \mathcal{S}$ is written as (2.3). We construct from v a process \tilde{v} in such a way that for each $w \in \mathbb{W}$,

$$\begin{aligned}\tilde{\xi}_0 &:= \xi_0, & \tilde{v}_t(w) &:= \tilde{\xi}_0 \text{ for } t_0 \leq t \leq t_1, \\ \tilde{\xi}_1(w) &:= \xi_1 \left(w(t) - \int_0^t \tilde{v}_s(w) ds, t \leq t_1 \right), & \tilde{v}_t(w) &:= \tilde{\xi}_1(w) \text{ for } t_1 < t \leq t_2, \\ \dots & & \dots & \\ \tilde{\xi}_{m-1}(w) &:= \xi_{m-1} \left(w(t) - \int_0^t \tilde{v}_s(w) ds, t \leq t_{m-1} \right), & \tilde{v}_t(w) &:= \tilde{\xi}_{m-1}(w) \text{ for } t_{m-1} < t \leq t_m,\end{aligned}$$

so that we have the relation

$$\tilde{v}(w) = v(T^{-\tilde{v}}(w)), \quad T^v \circ T^{-\tilde{v}}(w) = w\tag{2.9}$$

for all $w \in \mathbb{W}$. It is clear by construction that \tilde{v} is in \mathcal{S} , and hence in \mathcal{A}_b . Therefore by Girsanov's formula (2.2), the right-hand side of (2.8) is equal to

$$\begin{aligned}\mathbb{E}^{\tilde{v}} \left[F(T^v \circ T^{-\tilde{v}}(W)) - \frac{1}{2} \int_0^1 |v_s(T^{-\tilde{v}}(W))|^2 ds \right] \\ = \mathbb{E}^{\tilde{v}} \left[F(W) - \frac{1}{2} \int_0^1 |\tilde{v}_s|^2 ds \right],\end{aligned}$$

where we used (2.9) for the second line. By Lemma 2.3, the last expression is dominated by $\log \mathbb{E} [e^{F(W)}]$. This ends the proof. \square

Using Lemma 2.4, we prove

Lemma 2.5. *Let $F : \mathbb{W} \rightarrow \mathbb{R}$ be bounded and continuous. Then (2.8) holds for any $v \in \mathcal{A}$.*

Proof. Let $v \in \mathcal{A}$. By the density of \mathcal{S} in \mathcal{A} , there exists a sequence $\{v^n\}_{n \in \mathbb{N}} \subset \mathcal{S}$ such that $\|v^n - v\|_{\mathcal{A}} \rightarrow 0$ as $n \rightarrow \infty$. Then, since

$$\mathbb{E} \left[\left| \int_0^\cdot v_s^n ds - \int_0^\cdot v_s ds \right|_{\mathbb{W}}^2 \right] \leq \|v^n - v\|_{\mathcal{A}}^2 \xrightarrow{n \rightarrow \infty} 0,$$

we may extract a subsequence $\{n'\} \subset \mathbb{N}$ such that

$$\left| \int_0^\cdot v_s^{n'} ds - \int_0^\cdot v_s ds \right|_{\mathbb{W}} \xrightarrow{n' \rightarrow \infty} 0 \quad \text{a.s.} \quad (2.10)$$

Since each $v^{n'}$ is in \mathcal{S} and F is assumed to be bounded, we have by Lemma 2.4,

$$\log \mathbb{E} [e^{F(W)}] \geq \mathbb{E} \left[F \left(T^{v^{n'}}(W) \right) \right] - \frac{1}{2} \|v^{n'}\|_{\mathcal{A}}^2.$$

By (2.10) and the continuity of F , the bounded convergence theorem yields

$$\lim_{n' \rightarrow \infty} \mathbb{E} \left[F \left(T^{v^{n'}}(W) \right) \right] = \mathbb{E} [F(T^v(W))].$$

As $v^{n'}$ approximates v with respect to $\|\cdot\|_{\mathcal{A}}$, it also holds that $\|v^{n'}\|_{\mathcal{A}} \rightarrow \|v\|_{\mathcal{A}}$ as $n' \rightarrow \infty$. Combining these ends the proof. \square

Remark 2.2. In fact, when $\|v^n - v\|_{\mathcal{A}} \rightarrow 0$ as $n \rightarrow \infty$, the whole sequence $\{T^{v^n}(W)\}_{n \in \mathbb{N}}$ converges weakly to $T^v(W)$.

We stand ready to prove Lemma 2.2.

Proof of Lemma 2.2. Let $F : \mathbb{W} \rightarrow \mathbb{R}$ be bounded and measurable. Then there exists a sequence $\{F_n\}_{n \in \mathbb{N}}$ of bounded and continuous functions on \mathbb{W} such that

$$\lim_{n \rightarrow \infty} F_n = F \quad \text{a.s.} \quad \text{and} \quad \sup_{n \in \mathbb{N}} \|F_n\|_{\infty} \leq \|F\|_{\infty} \quad (2.11)$$

as is recalled in [4, Theorem 2.6] from [10, Theorem V.16 (a)]. Take $v \in \mathcal{A}$ arbitrarily. Then by Lemma 2.5, we have for every $n \in \mathbb{N}$,

$$\log \mathbb{E} [e^{F_n(W)}] \geq \mathbb{E} [F_n(T^v(W))] - \frac{1}{2} \|v\|_{\mathcal{A}}^2.$$

The left-hand side converges to $\log \mathbb{E} [e^{F(W)}]$ as $n \rightarrow \infty$ by (2.11) and the bounded convergence theorem. Moreover, we also have

$$\lim_{n \rightarrow \infty} \mathbb{E} [F_n(T^v(W))] = \mathbb{E} [F(T^v(W))]$$

since the law $\mathbb{P} \circ (T^v)^{-1}$ is absolutely continuous with respect to \mathbb{P} (see [18, Theorem 4] and [19, Theorem 7.4]) thanks to $\int_0^1 |v_s|^2 ds < \infty$, \mathbb{P} -a.s. Combining these leads to the conclusion. \square

Remark 2.3. The above-mentioned absolute continuity may also be inferred from the finiteness of the relative entropy of $\mathbb{P} \circ (T^v)^{-1}$ with respect to \mathbb{P} , shown in equation (12) of [4].

2.2 Proof of the upper bound

In this subsection we prove the upper bound in (1.2):

Proposition 2.2. *Assume that a measurable function $F : \mathbb{W} \rightarrow \mathbb{R}$ satisfies (A1) and (A2). Then it holds that*

$$\log \mathbb{E} [e^{F(W)}] \leq \sup_{v \in \mathcal{A}} \left\{ \mathbb{E} [F(T^v(W))] - \frac{1}{2} \|v\|_{\mathcal{A}}^2 \right\}. \quad (2.12)$$

Using the notion of filtration introduced by Üstünel and Zakai [28] on an abstract Wiener space, Zhang [29] extended the variational representation (1.1) of Boué-Dupuis for bounded Wiener functionals to the framework of abstract Wiener spaces as simplifying considerably the original proof of the upper bound by employing the Clark-Ocone formula. We also make use of the Clark-Ocone formula to prove Proposition 2.2.

First we prove (2.12) in the case that F satisfies (A2) and is bounded from above:

$$M \equiv M_F := \operatorname{ess\,sup}_{w \in \mathbb{W}} F(w) < \infty. \quad (2.13)$$

We denote by \mathcal{FC}_b^1 the set of all functionals on \mathbb{W} of the form

$$f(w(t_1), \dots, w(t_m)), \quad w \in \mathbb{W}, \quad (2.14)$$

for some $m \in \mathbb{N}$, $0 \leq t_1 < \dots < t_m \leq 1$ and for some bounded C^1 -function $f : (\mathbb{R}^d)^m \rightarrow \mathbb{R}$ whose partial derivatives are all bounded as well. Since F is in $L^1(\mathbb{P})$ by the assumption (A2) and (2.13), we may find a sequence $\{F_n\}_{n \in \mathbb{N}} \subset \mathcal{FC}_b^1$ such that

$$\lim_{n \rightarrow \infty} \mathbb{E} [|F_n(W) - F(W)|] = 0. \quad (2.15)$$

Truncating F_n if necessary, we may moreover assume that

$$\sup_{n \in \mathbb{N}} \sup_{w \in \mathbb{W}} F_n(w) \leq M. \quad (2.16)$$

We fix such a sequence $\{F_n\}_{n \in \mathbb{N}}$. The following lemma is immediate from the Clark-Ocone formula and Itô's formula.

Lemma 2.6. *For each $n \in \mathbb{N}$, there exists $v^n \in \mathcal{A}_b$ such that*

$$\frac{\mathbb{E} [e^{F_n(W)} | \mathcal{F}_t]}{\mathbb{E} [e^{F_n(W)}]} = \mathcal{E}_t^{v^n} \quad a.s. \quad (2.17)$$

for all $0 \leq t \leq 1$.

In fact, if F_n is written as (2.14), then the claimed v^n admits the expression

$$v_t^n = \sum_{k=1}^m \mathbf{1}_{[0, t_k]}(t) \frac{\mathbb{E} [e^{F_n(W)} \nabla_{x^k} f(W(t_1), \dots, W(t_m)) | \mathcal{F}_t]}{\mathbb{E} [e^{F_n(W)} | \mathcal{F}_t]} \quad a.s.$$

for all $0 \leq t \leq 1$. For the Clark-Ocone formula, we refer the reader to [17, Appendix E] and [20, Proposition 1.3.14].

Lemma 2.7. *Let $\{v^n\}_{n \in \mathbb{N}} \subset \mathcal{A}_b$ be as given in Lemma 2.6. Then for each $n \in \mathbb{N}$, we have*

$$\log \mathbb{E} [e^{F_n(W)}] = \mathbb{E}^{v^n} \left[F_n(W) - \frac{1}{2} \int_0^1 |v_s^n|^2 ds \right].$$

Proof. When $t = 1$ we rewrite (2.17) in such a way that

$$\log \mathbb{E} [e^{F_n(W)}] = F_n(W) - \log \mathcal{E}_1^{v^n} \quad \mathbb{P}\text{-a.s.}$$

Taking the \mathbb{P}^{v^n} -expectation on the right-hand side and recalling the identity (2.7), we have the lemma. \square

Using this lemma, we divide the left-hand side of (2.12) into three parts

$$\log \mathbb{E} [e^{F(W)}] = I_n^1 + I_n^2 + I_n^3 \quad (2.18)$$

for each $n \in \mathbb{N}$, where we set

$$\begin{aligned} I_n^1 &= \log \mathbb{E} [e^{F(W)}] - \log \mathbb{E} [e^{F_n(W)}], \\ I_n^2 &= \mathbb{E}^{v^n} [F_n(W) - F(W)], \\ I_n^3 &= \mathbb{E}^{v^n} \left[F(W) - \frac{1}{2} \int_0^1 |v_s^n|^2 ds \right]. \end{aligned}$$

Note that this decomposition makes sense because Hölder's inequality yields

$$\mathbb{E} [F_-(W) \mathcal{E}_1^{v^n}] \leq \|F_-(W)\|_{1+\delta} \|\mathcal{E}_1^{v^n}\|_{1+1/\delta} < \infty$$

by the assumption (A2) and Lemma 2.1.

Lemma 2.8. *We have*

$$\lim_{n \rightarrow \infty} I_n^i = 0, \quad i = 1, 2. \quad (2.19)$$

Proof. Since the function $\mathbb{R} \ni z \mapsto e^z$ is increasing and convex, we have $|e^{z_1} - e^{z_2}| \leq e^M |z_1 - z_2|$ for any $z_1, z_2 \leq M$, and hence by (2.13) and (2.16),

$$|\mathbb{E} [e^{F(W)}] - \mathbb{E} [e^{F_n(W)}]| \leq e^M \mathbb{E} [|F(W) - F_n(W)|]$$

for all $n \in \mathbb{N}$. This implies (2.19) for $i = 1$ by (2.15).

As for I_n^2 , we fix an $\varepsilon > 0$. By (2.15), for all sufficiently large n ,

$$\mathbb{E} [F_n(W)] \geq \mathbb{E} [F(W)] - \varepsilon,$$

hence by Jensen's inequality,

$$\mathbb{E} [e^{F_n(W)}] \geq \exp (\mathbb{E} [F(W)] - \varepsilon).$$

By this estimate, Lemma 2.6 and (2.16), we have

$$\begin{aligned}\mathcal{E}_1^{v^n} &= \frac{e^{F_n(W)}}{\mathbb{E}[e^{F_n(W)}]} \\ &\leq \exp(M + \varepsilon - \mathbb{E}[F(W)])\end{aligned}$$

if n is sufficiently large. Then by the definition of (2.1) of \mathbb{P}^{v^n} ,

$$|I_n^2| \leq \exp(M + \varepsilon - \mathbb{E}[F(W)]) \mathbb{E}[|F_n(W) - F(W)|],$$

which tends to 0 as $n \rightarrow \infty$ by (2.15). The proof is complete. \square

As for I_n^3 , we have the estimate

$$\sup_{n \in \mathbb{N}} I_n^3 \leq \sup_{v \in \mathcal{A}} \left\{ \mathbb{E}[F(T^v(W))] - \frac{1}{2} \|v\|_{\mathcal{A}}^2 \right\}, \quad (2.20)$$

proof of which is postponed to Subsection 2.3. Putting (2.18), (2.19) and (2.20) together, we have now arrived at

Proposition 2.3. *The upper bound (2.12) holds for any measurable function $F : \mathbb{W} \rightarrow \mathbb{R}$ that satisfies (A2) and is bounded from above.*

Proof. By (2.18) and (2.20), we have

$$\log \mathbb{E}[e^{F(W)}] \leq I_n^1 + I_n^2 + \sup_{v \in \mathcal{A}} \left\{ \mathbb{E}[F(T^v(W))] - \frac{1}{2} \|v\|_{\mathcal{A}}^2 \right\}$$

for all $n \in \mathbb{N}$. By letting $n \rightarrow \infty$, the assertion follows from Lemma 2.8. \square

The proof of Proposition 2.2 is immediate from Proposition 2.3.

Proof of Proposition 2.2. For a measurable function $F : \mathbb{W} \rightarrow \mathbb{R}$ satisfying (A1) and (A2), we set for each $N > 0$,

$$F_N(w) := F(w) \wedge N, \quad w \in \mathbb{W}.$$

Then for any N , we have by Proposition 2.3,

$$\begin{aligned}\log \mathbb{E}[e^{F_N(W)}] &\leq \sup_{v \in \mathcal{A}} \left\{ \mathbb{E}[F_N(T^v(W))] - \frac{1}{2} \|v\|_{\mathcal{A}}^2 \right\} \\ &\leq \sup_{v \in \mathcal{A}} \left\{ \mathbb{E}[F(T^v(W))] - \frac{1}{2} \|v\|_{\mathcal{A}}^2 \right\}.\end{aligned}$$

Letting $N \rightarrow \infty$ on the leftmost side leads to the conclusion by the dominated convergence theorem. \square

2.3 Proof of (2.20)

In this subsection we prove the estimate (2.20), which we rephrase in a slightly stronger statement that

Proposition 2.4. *Let $F : \mathbb{W} \rightarrow \mathbb{R}$ be a measurable function satisfying (A2) and (2.13). Then it holds that for any $v \in \mathcal{A}_b$,*

$$\mathbb{E}^v \left[F(W) - \frac{1}{2} \int_0^1 |v_s|^2 ds \right] \leq \sup_{\bar{v} \in \mathcal{S}} \left\{ \mathbb{E} [F(T^{\bar{v}}(W))] - \frac{1}{2} \|\bar{v}\|_{\mathcal{A}}^2 \right\}. \quad (2.21)$$

A key to the proof of this proposition is Lemma 2.9 below, which is an immediate consequence of Scheffé's lemma. We note that Scheffé's lemma is also employed in Osuka [21], where the variational representation (1.1) of Boué-Dupuis is extended to bounded functionals of G -Brownian motion, an extended notion of Brownian motion introduced by Peng [22, 23], to the framework of sublinear expectation spaces.

Fix $v \in \mathcal{A}_b$ and let $\{v^n\}_{n \in \mathbb{N}} \subset \mathcal{S}$ be such that

$$\lim_{n \rightarrow \infty} \|v^n - v\|_{\mathcal{A}} = 0 \quad (2.22)$$

and that by truncating each v^n if necessary,

$$\sup_{n \in \mathbb{N}} \sup_{0 \leq t \leq 1} \|v_t^n\|_{\infty} \leq \sup_{0 \leq t \leq 1} \|v_t\|_{\infty} =: K < \infty. \quad (2.23)$$

Note that such an approximate sequence exists by the density of \mathcal{S} in \mathcal{A} .

Lemma 2.9. *It holds that*

$$\lim_{n \rightarrow \infty} \|\mathcal{E}_1^{v^n} - \mathcal{E}_1^v\|_1 = 0. \quad (2.24)$$

Proof. Fix an arbitrary subsequence $\{n'\} \subset \mathbb{N}$. It suffices to prove the existence of a subsequence of $\{n'\}$ along which the convergence (2.24) takes place. By Itô's isometry and (2.22),

$$\mathbb{E} \left[\left| \int_0^1 v_s^{n'} \cdot dW_s - \int_0^1 v_s \cdot dW_s \right|^2 \right] = \|v^{n'} - v\|_{\mathcal{A}}^2 \xrightarrow{n' \rightarrow \infty} 0.$$

Moreover, we have by (2.22) and (2.23),

$$\mathbb{E} \left[\left| \int_0^1 |v_s^{n'}|^2 ds - \int_0^1 |v_s|^2 ds \right| \right] \leq 2K \mathbb{E} \left[\int_0^1 |v_s^{n'} - v_s| ds \right] \xrightarrow{n' \rightarrow \infty} 0.$$

Therefore we may extract a subsequence $\{n''\} \subset \{n'\}$ such that

$$\lim_{n'' \rightarrow \infty} \left(\int_0^1 v_s^{n''} \cdot dW_s - \frac{1}{2} \int_0^1 |v_s^{n''}|^2 ds \right) = \int_0^1 v_s \cdot dW_s - \frac{1}{2} \int_0^1 |v_s|^2 ds \quad \text{a.s.} \quad (2.25)$$

By the boundedness of $v^{n''}$ and v , Novikov's condition entails that

$$\mathbb{E}[\mathcal{E}_1^{v^{n''}}] = \mathbb{E}[\mathcal{E}_1^v] = 1 \quad \text{for all } n''. \quad (2.26)$$

By (2.25), (2.26) and Scheffé's lemma, we conclude that the convergence (2.24) takes place along $\{n''\}$. This proves the lemma. \square

For every $n \in \mathbb{N}$, we decompose the left-hand side of (2.21) into the sum

$$\mathbb{E}^v \left[F(W) - \frac{1}{2} \int_0^1 |v_s|^2 ds \right] = J_n^1 + \frac{1}{2} J_n^2 + J_n^3, \quad (2.27)$$

where we set

$$\begin{aligned} J_n^1 &= \mathbb{E} [F(W)(\mathcal{E}_1^v - \mathcal{E}_1^{v^n})], \\ J_n^2 &= \mathbb{E}^{v^n} \left[\int_0^1 |v_s^n|^2 ds \right] - \mathbb{E}^v \left[\int_0^1 |v_s|^2 ds \right], \\ J_n^3 &= \mathbb{E}^{v^n} \left[F(W) - \frac{1}{2} \int_0^1 |v_s^n|^2 ds \right]. \end{aligned}$$

Lemma 2.10. *We have*

$$\lim_{n \rightarrow \infty} J_n^i = 0, \quad i = 1, 2. \quad (2.28)$$

Proof. Fix $N > 0$ arbitrarily. Observe the bound

$$\begin{aligned} |J_n^1| &\leq \mathbb{E} [|F(W) \mathbf{1}_{\{F(W) > -N\}}| |\mathcal{E}_1^v - \mathcal{E}_1^{v^n}|] + \mathbb{E} [|F(W) \mathbf{1}_{\{F(W) \leq -N\}}| |\mathcal{E}_1^v - \mathcal{E}_1^{v^n}|] \\ &\leq (M \vee N) \|\mathcal{E}_1^v - \mathcal{E}_1^{v^n}\|_1 + C \|F_-(W) \mathbf{1}_{\{F_-(W) \geq N\}}\|_{1+\delta} \end{aligned} \quad (2.29)$$

with $C := \sup_{n \in \mathbb{N}} \|\mathcal{E}_1^v - \mathcal{E}_1^{v^n}\|_{1+1/\delta} < \infty$, where for the second line we used (2.13) and Hölder's inequality; the finiteness of C is due to (2.23) and Lemma 2.1. Letting $n \rightarrow \infty$ on both sides of (2.29), we see from Lemma 2.9 that

$$\limsup_{n \rightarrow \infty} |J_n^1| \leq C \|F_-(W) \mathbf{1}_{\{F_-(W) \geq N\}}\|_{1+\delta}$$

for any $N > 0$. Since the right-hand side tends to 0 as $N \rightarrow \infty$ by the assumption (A2), we obtain (2.28) for $i = 1$.

As for J_n^2 , we observe that by (2.23),

$$\begin{aligned} |J_n^2| &\leq \mathbb{E} \left[\int_0^1 |v_s^n|^2 ds |\mathcal{E}_1^{v^n} - \mathcal{E}_1^v| \right] + \mathbb{E} \left[\left| \int_0^1 (|v_s^n|^2 - |v_s|^2) ds \right| \mathcal{E}_1^v \right] \\ &\leq K^2 \|\mathcal{E}_1^{v^n} - \mathcal{E}_1^v\|_1 + 2K \|v^n - v\|_{\mathcal{A}} \|\mathcal{E}_1^v\|_2. \end{aligned}$$

Since $\|\mathcal{E}_1^v\|_2 < \infty$ by Lemma 2.1, the last expression tends to 0 as $n \rightarrow \infty$ by Lemma 2.9 and (2.22). The proof of the lemma is complete. \square

Concerning J_n^3 , we have

Lemma 2.11. *It holds that*

$$\sup_{n \in \mathbb{N}} J_n^3 \leq \sup_{\bar{v} \in \mathcal{S}} \left\{ \mathbb{E} [F(T^{\bar{v}}(W))] - \frac{1}{2} \|\bar{v}\|_{\mathcal{A}}^2 \right\}.$$

Proof. Fix $n \in \mathbb{N}$. Since v^n is in \mathcal{S} , we may represent v^n as (2.3). We construct from v^n a process \bar{v} in such a way that for each $w \in \mathbb{W}$,

$$\begin{aligned} \bar{\xi}_0 &:= \xi_0, & \bar{v}_t(w) &:= \bar{\xi}_0 \text{ for } t_0 \leq t \leq t_1, \\ \bar{\xi}_1(w) &:= \xi_1 \left(w(t) + \int_0^t \bar{v}_s(w) ds, t \leq t_1 \right), & \bar{v}_t(w) &:= \bar{\xi}_1(w) \text{ for } t_1 < t \leq t_2, \\ \dots & & \dots & \\ \bar{\xi}_{m-1}(w) &:= \xi_{m-1} \left(w(t) + \int_0^t \bar{v}_s(w) ds, t \leq t_{m-1} \right), & \bar{v}_t(w) &:= \bar{\xi}_{m-1}(w) \text{ for } t_{m-1} < t \leq t_m. \end{aligned}$$

Note that \bar{v} is in \mathcal{S} by construction; moreover, by induction on $k = 1, \dots, m$, we have for all $w \in \mathbb{W}$,

$$v_t^n(w) = \bar{v}_t(T^{-v^n}(w)), \quad 0 \leq t \leq t_k, \quad k = 1, \dots, m, \quad (2.30)$$

from which it also follows that

$$T^{\bar{v}} \circ T^{-v^n}(w) = w \quad \text{for all } w \in \mathbb{W}. \quad (2.31)$$

These relations were noticed in Zhang [29]. Using (2.30) and (2.31), we rewrite J_n^3 as

$$\begin{aligned} J_n^3 &= \mathbb{E}^{v^n} \left[F(T^{\bar{v}} \circ T^{-v^n}(W)) - \frac{1}{2} \int_0^1 |\bar{v}_s(T^{-v^n}(W))|^2 ds \right] \\ &= \mathbb{E} \left[F(T^{\bar{v}}(W)) - \frac{1}{2} \int_0^1 |\bar{v}_s|^2 ds \right], \end{aligned}$$

where the second line follows from the boundedness from above of F and Girsanov's formula (2.2). Since $\bar{v} \in \mathcal{S}$, the lemma is proven. \square

We are in a position to prove Proposition 2.4.

Proof of Proposition 2.4. By (2.27) and Lemma 2.11, we have

$$\mathbb{E}^v \left[F(W) - \frac{1}{2} \int_0^1 |v_s|^2 ds \right] \leq J_n^1 + \frac{1}{2} J_n^2 + \sup_{\bar{v} \in \mathcal{S}} \left\{ \mathbb{E} [F(T^{\bar{v}}(W))] - \frac{1}{2} \|\bar{v}\|_{\mathcal{A}}^2 \right\}$$

for all $n \in \mathbb{N}$. The assertion follows by letting $n \rightarrow \infty$ thanks to Lemma 2.10. \square

Proposition 2.4 reveals that we may replace the supremum over $v \in \mathcal{A}$ in the variational representation (1.2) by that over $v \in \mathcal{S}$; by adopting the convention that $\log \infty = \infty$, the representation (1.2) with this replacement remains true even if we remove the assumption (A1). For later use, we state it in a proposition.

Proposition 2.5. *Let $F : \mathbb{W} \rightarrow \mathbb{R}$ be measurable and satisfy (A2). Then it holds that*

$$\log \mathbb{E} [e^{F(W)}] = \sup_{v \in \mathcal{S}} \left\{ \mathbb{E} [F(T^v(W))] - \frac{1}{2} \|v\|_{\mathcal{A}}^2 \right\}, \quad (2.32)$$

where the left-hand side is understood to be equal to ∞ when $\mathbb{E} [e^{F(W)}] = \infty$.

Proof. For every $M > 0$, we set $F_M(w) := F(w) \wedge M$, $w \in \mathbb{W}$. By Propositions 2.1 and 2.4 together with the proof of Proposition 2.3, we see that (2.32) holds for F_M :

$$\log \mathbb{E} [e^{F_M(W)}] = \sup_{v \in \mathcal{S}} \left\{ \mathbb{E} [F_M(T^v(W))] - \frac{1}{2} \|v\|_{\mathcal{A}}^2 \right\}. \quad (2.33)$$

Letting $M \rightarrow \infty$, we have the convergence of the left-hand side to $\log \mathbb{E} [e^{F(W)}]$ by the monotone convergence theorem. As for the right-hand side, note that

$$\mathbb{E} [F_-(T^v(W))] < \infty \quad \text{for each } v \in \mathcal{S};$$

indeed, constructing from v a process \tilde{v} in \mathcal{S} that satisfies the relation (2.9), we have

$$\begin{aligned} \mathbb{E} [F_-(T^v(W))] &= \mathbb{E}^{\tilde{v}} [F_-(T^v \circ T^{-\tilde{v}}(W))] \\ &= \mathbb{E} [F_-(W) \mathcal{E}_1^{\tilde{v}}] \\ &\leq \|F_-(W)\|_{1+\delta} \|\mathcal{E}_1^{\tilde{v}}\|_{1+1/\delta}, \end{aligned}$$

which is finite by (A2), the boundedness of \tilde{v} and Lemma 2.1. Here we used Girsanov's formula (2.2) for the first line, the relation (2.9) and the definition (2.1) of $\mathbb{P}^{\tilde{v}}$ for the second, and Hölder's inequality for the third. Therefore we may also apply the monotone convergence theorem to the right-hand side of (2.33) to obtain

$$\sup_{M>0} \sup_{v \in \mathcal{S}} \left\{ \mathbb{E} [F_M(T^v(W))] - \frac{1}{2} \|v\|_{\mathcal{A}}^2 \right\} = \sup_{v \in \mathcal{S}} \left\{ \mathbb{E} [F(T^v(W))] - \frac{1}{2} \|v\|_{\mathcal{A}}^2 \right\}$$

(cf. (1.3)), which concludes the proof. \square

We end this section with a remark on the proof of Theorem 1.1.

Remark 2.4. (1) The above proof is also valid in the setting of an abstract Wiener space; we may extend the variational representation of Zhang [29] for bounded functionals on the abstract Wiener space to functionals satisfying conditions corresponding to (A1) and (A2).

(2) Since both sides of (1.2) are well-defined only under the assumption (A1) as noted in Remark 1.1 (2), it seems plausible that the representation (1.2) holds true without any assumptions on F from below; however, we have not succeeded in proving it. The problem is how to prove the upper bound (2.12) without the integrability assumption (A2).

3 Prékopa's theorem on Wiener space and its application

In this section we prove Theorem 1.2 and provide its application in Theorem 3.1, which extends the Brascamp-Lieb inequality to the Wiener space.

3.1 Proof of Theorem 1.2

In this subsection we give a proof of Theorem 1.2 as an immediate application of Proposition 2.5.

Proof of Theorem 1.2. Set $g(\lambda) = \log \mathbb{E} [e^{G(W, \lambda)}]$, $\lambda \in \Lambda$. We fix $\lambda_1, \lambda_2 \in \Lambda$ and $\theta \in [0, 1]$ arbitrarily. For any $v^1, v^2 \in \mathcal{S}$, we have by the condition (B2),

$$\begin{aligned} & G(T^{\theta v^1 + (1-\theta)v^2}(w), \theta\lambda_1 + (1-\theta)\lambda_2) - \frac{1}{2} \int_0^1 |\theta v_s^1(w) + (1-\theta)v_s^2(w)|^2 ds \\ & \geq \theta \left\{ G(T^{v^1}(w), \lambda_1) - \frac{1}{2} \int_0^1 |v_s^1(w)|^2 ds \right\} + (1-\theta) \left\{ G(T^{v^2}(w), \lambda_2) - \frac{1}{2} \int_0^1 |v_s^2(w)|^2 ds \right\} \end{aligned}$$

for all $w \in \mathbb{W}$. Noting $\theta v^1 + (1-\theta)v^2 \in \mathcal{S}$, we take the expectation in w with respect to \mathbb{P} on both sides to get

$$\begin{aligned} g(\theta\lambda_1 + (1-\theta)\lambda_2) &= \sup_{v \in \mathcal{S}} \left\{ \mathbb{E} [G(T^v(W), \theta\lambda_1 + (1-\theta)\lambda_2)] - \frac{1}{2} \|v\|_{\mathcal{A}}^2 \right\} \\ &\geq \theta \left\{ \mathbb{E} [G(T^{v^1}(W), \lambda_1)] - \frac{1}{2} \|v^1\|_{\mathcal{A}}^2 \right\} \\ &\quad + (1-\theta) \left\{ \mathbb{E} [G(T^{v^2}(W), \lambda_2)] - \frac{1}{2} \|v^2\|_{\mathcal{A}}^2 \right\} \end{aligned}$$

for any $v^1, v^2 \in \mathcal{S}$. Here the equality is due to (B1) and Proposition 2.5. Maximizing the rightmost side over v^1 and v^2 , and using Proposition 2.5, we obtain

$$g(\theta\lambda_1 + (1-\theta)\lambda_2) \geq \theta g(\lambda_1) + (1-\theta)g(\lambda_2)$$

as claimed. □

Remark 3.1. (1) In [26, Subsection 13.A], some convexity results are shown as to the Schrödinger operator $-(1/2)\Delta + V$ in \mathbb{R}^d with V a convex function, such as the log-concavity of its ground state and the convexity of the infimum of its spectrum relative to an additional parameter put into the operator; these are derived by employing the time discretization of the associated Feynman-Kac path integral representations and finite-dimensional Prékopa's theorem. We can also prove those results by using Theorem 1.2; the advantage is that discretization procedures are not required at all.

(2) Theorem 1.2 can also be extended to the framework of abstract Wiener spaces.

3.2 Brascamp-Lieb inequality on Wiener space

In this subsection, the Brascamp-Lieb inequality formulated on the Wiener space is shown as an application of Theorem 1.2.

We denote by \mathbb{W}^* (resp. \mathbb{H}^*) the topological dual space of \mathbb{W} (resp. of \mathbb{H}) and by $\langle \cdot, \cdot \rangle \equiv {}_{\mathbb{W}^*} \langle \cdot, \cdot \rangle_{\mathbb{W}}$ the natural coupling between \mathbb{W}^* and \mathbb{W} . Identifying \mathbb{H}^* with \mathbb{H} and noting the inclusion $\mathbb{W}^* \subset \mathbb{H}^*$, we regard each $l \in \mathbb{W}^*$ as an element in \mathbb{H} , which we still denote by l .

Theorem 3.1. *Let $F : \mathbb{W} \rightarrow \mathbb{R}$ be a measurable function satisfying the following assumptions (C1)–(C3):*

(C1) *F is concave on \mathbb{W} ;*

(C2) $\mathbb{E} [e^{F(W)}] < \infty$;

(C3) *there exists $\delta > 0$ such that $F_- \in L^{1+\delta}(\mathbb{P})$.*

We define the probability measure \mathbb{Q} on $(\mathbb{W}, \mathcal{B}(\mathbb{W}))$ by

$$\mathbb{Q}(A) := \frac{\mathbb{E} [\mathbf{1}_A e^{F(W)}]}{\mathbb{E} [e^{F(W)}]}, \quad A \in \mathcal{B}(\mathbb{W}),$$

and denote by $\mathbb{E}_{\mathbb{Q}}$ the expectation with respect to \mathbb{Q} . Then it holds that for any nonzero $l \in \mathbb{W}^*$ and for any convex function ψ on \mathbb{R} ,

$$\mathbb{E}_{\mathbb{Q}} [\psi(\langle l, W \rangle - \mathbb{E}_{\mathbb{Q}} [\langle l, W \rangle])] \leq \frac{1}{\sqrt{2\pi}|l|_{\mathbb{H}}} \int_{\mathbb{R}} \psi(z) \exp\left(-\frac{z^2}{2|l|_{\mathbb{H}}^2}\right) dz. \quad (3.1)$$

Remark 3.2. Suppose that $F : \mathbb{W} \rightarrow \mathbb{R}$ satisfies (C1) and is upper-semicontinuous on \mathbb{W} . Then the assumption (C2) is fulfilled because F is bounded from above by an affine function [1, Proposition 2.20]; we also refer to the fact that the concavity and upper-semicontinuity of F yield the continuity of F [1, Proposition 2.16] as \mathbb{W} is a Banach space.

Let $V : \mathbb{R}^d \rightarrow \mathbb{R}$ be a convex function and Σ a symmetric, positive definite $d \times d$ -matrix. We consider the case that F is given by

$$F(w) = -V(\Sigma^{1/2}w(1)), \quad w \in \mathbb{W},$$

and satisfies (C3), and that l is of the form

$$\langle l, w \rangle = \Sigma^{1/2}\alpha \cdot w(1), \quad w \in \mathbb{W},$$

for a given $\alpha \in \mathbb{R}^d$ ($\alpha \neq 0$). In this case the inequality (3.1) is restated as

$$E[\psi(\alpha \cdot X - E[\alpha \cdot X])] \leq E[\psi(\alpha \cdot Y)] \quad (3.2)$$

for any convex function ψ on \mathbb{R} . Here X and Y are \mathbb{R}^d -valued random variables defined on a probability space (Ω, \mathcal{F}, P) , whose laws induced on \mathbb{R}^d are given respectively by

$$P(X \in dx) = \frac{1}{Z} e^{-V(x)} \nu(dx), \quad P(Y \in dx) = \nu(dx), \quad (3.3)$$

where ν is the normal distribution with mean 0 and covariance matrix Σ and Z is the normalizing constant. The inequality (3.2) is referred to as the Brascamp-Lieb (moment) inequality; it was originally proven by Brascamp and Lieb [6, Theorem 5.1] in the case $\psi(z) = |z|^p$, $p \geq 1$, and later extended by Caffarelli [8, Corollary 6] to general convex ψ 's based on analyses of the optimal transport between the laws of X and Y . In [14], the author gives a proof of the Brascamp-Lieb inequality (3.2) based on the Skorokhod embedding and the Itô-Tanaka formula, and derives error estimates for the inequality in terms of the variances of $\alpha \cdot X$ and $\alpha \cdot Y$ [14, Theorem 1.1]. In these three papers proofs of (3.2) are reduced to the one-dimensional case thanks to finite-dimensional Prékopa's theorem. The proof of Theorem 3.1 is done in the same way by employing Theorem 1.2, an infinite-dimensional version of Prékopa's theorem.

Proof of Theorem 3.1. Fix $l \in \mathbb{W}^*$ ($l \neq 0$). We may assume without loss of generality that $|l|_{\mathbb{H}} = 1$. Since the law of $\langle l, W \rangle$ under \mathbb{Q} is expressed as

$$\mathbb{Q}(\langle l, W \rangle \in dz) = \frac{1}{\sqrt{2\pi} \mathbb{E}[e^{F(W)}]} \exp\left(-\frac{z^2}{2}\right) \mathbb{E}[e^{F(W)} | \langle l, W \rangle = z] dz, \quad z \in \mathbb{R},$$

it suffices to prove that the function

$$\mathbb{R} \ni z \mapsto \mathbb{E}[e^{F(W)} | \langle l, W \rangle = z] \quad (3.4)$$

admits an everywhere finite version that is log-concave in z . To this end, define the path transform w^l , $w \in \mathbb{W}$, by

$$w^l(t) := w(t) - \langle l, w \rangle l(t), \quad 0 \leq t \leq 1, \quad (3.5)$$

where l is regarded as an element in \mathbb{H} . Since two Gaussians $\{W^l(t)\}_{0 \leq t \leq 1}$ and $\langle l, W \rangle$ are uncorrelated, they are independent, from which we have

$$\mathbb{E}[e^{F(W)} | \langle l, W \rangle = z] = \mathbb{E}[e^{G(w, z)}] \quad \text{for a.e. } z \in \mathbb{R}, \quad (3.6)$$

where we set $G(w, z) := F(w^l + zl)$, $(w, z) \in \mathbb{W} \times \mathbb{R}$. In view of Theorem 1.2, we show that this G satisfies the conditions (B1) and (B2). It is clear that G satisfies (B2) thanks to the concavity of F and the linearity of the transformation (3.5). To see that (B1) is fulfilled, first note that by the assumption (C3),

$$\mathbb{E}[G_-(W, z)^{1+\delta}] < \infty \quad (3.7)$$

for a.e. $z \in \mathbb{R}$, which readily follows by conditioning on $\langle l, W \rangle$ and using the independence noted above. We now show that this a.e. finiteness can be extended to

the everywhere finiteness. For this purpose we fix $z_0 \in \mathbb{R}$ arbitrarily. Then we may find $z_i \in \mathbb{R}$, $i = 1, 2$, such that $z_1 < z_0 < z_2$ and that each z_i satisfies (3.7). Since the function $\mathbb{R} \ni r \mapsto (r \vee 0)^{1+\delta}$ is convex and nondecreasing, and the function $\mathbb{R} \ni z \mapsto -G(w, z)$ is convex for every fixed $w \in \mathbb{W}$, their composition, namely $G_-(w, z)^{1+\delta} = ((-G(w, z)) \vee 0)^{1+\delta}$, is also convex in z , which entails that

$$G_-(w, z_0)^{1+\delta} \leq \theta G_-(w, z_1)^{1+\delta} + (1 - \theta) G_-(w, z_2)^{1+\delta}$$

for every $w \in \mathbb{W}$. Here $\theta = (z_2 - z_0)/(z_2 - z_1) \in (0, 1)$. Taking the expectation in w with respect to \mathbb{P} on both sides and noting the finiteness (3.7) for z_i , $i = 1, 2$, we obtain

$$\mathbb{E} [G_-(W, z_0)^{1+\delta}] < \infty.$$

As $z_0 \in \mathbb{R}$ is arbitrary, this shows that G satisfies the condition (B1) as well. Therefore by Theorem 1.2, the function $\mathbb{R} \ni z \mapsto \log \mathbb{E} [e^{G(W, z)}]$ is concave. This function might take the value ∞ , but is finite a.e. by the assumption (C2) and the relation (3.6), which together with concavity implies that it is in fact finite everywhere. Consequently, the function (3.4) admits the everywhere finite log-concave version $\mathbb{E} [e^{G(W, z)}]$. The rest of the proof of the theorem proceeds in the same way as in either [6], [8], or [14]. \square

Appendix

In this appendix, we continue our discussion in [14, Appendix] as to an extension of the Brascamp-Lieb inequality (3.2) to the case of nonconvex potentials and explore conditions on the potential function V under which the inequality (3.2) remains true. We restrict our exposition to one dimension; a remark on the multidimensional case will be given at the end of the appendix. The Brascamp-Lieb inequality has importance in the analysis of $\nabla\phi$ interface models with convex potentials and there has recently been growing a great interest in models with nonconvex potentials; see [12, 13, 3, 9] and references therein.

Let ν be the normal distribution with mean 0 and variance σ^2 , $\sigma > 0$, and let one-dimensional random variables X and Y be as given in (3.3), in which we now suppose that the function $V : \mathbb{R} \rightarrow \mathbb{R}$ is in $C^2(\mathbb{R})$ and not necessarily convex. We assume that V is bounded from below by a linear function:

$$V(x) \geq ax + b \quad \text{for all } x \in \mathbb{R}, \tag{A.1}$$

for some $a, b \in \mathbb{R}$, so that

$$Z = E [e^{-V(Y)}] < \infty.$$

We are interested in the case that $\{x \in \mathbb{R}; V''(x) < 0\} \neq \emptyset$, which we will work in from now on. We denote

$$\mathcal{D}_V = \{x \in \mathbb{R}; V''(x) \leq 0\}.$$

With these settings, the aim of this appendix is to give a proof of the

Proposition A.1. *Suppose that*

$$\inf_{x \in \mathcal{D}_V} \left\{ \frac{1}{2} \sigma^2 V'(x)^2 + x V'(x) - V(x) \right\} \geq \log Z. \quad (\text{A.2})$$

Then it holds that for any convex function ψ on \mathbb{R} ,

$$E[\psi(X - E[X])] \leq E[\psi(Y)]. \quad (\text{A.3})$$

In particular, the same conclusion holds true if

$$\inf_{x \in \mathcal{D}_V} \left\{ -\frac{x^2}{2\sigma^2} - V(x) \right\} \geq \log Z. \quad (\text{A.4})$$

We give an example:

Example A.1. Consider the potential V of the form

$$V(x) = \frac{1}{2} \alpha^2 x^4 - \frac{1}{2} \beta x^2, \quad x \in \mathbb{R},$$

for $\alpha, \beta > 0$. Take $\sigma = 1$ for simplicity. Then the left-hand side of (A.4) is calculated as

$$\frac{\beta(5\beta - 6)}{72\alpha^2} \wedge 0,$$

which tends to 0 as $\alpha \rightarrow \infty$. On the other hand, as

$$Z = \frac{1}{\sqrt{2\pi\alpha}} \int_{\mathbb{R}} \exp\left(\frac{\beta - 1}{2\alpha} y^2 - \frac{1}{2} y^4\right) dy$$

by change of variables, it is clear that the right-hand side of (A.4) diverges to $-\infty$ as $\alpha \rightarrow \infty$. Therefore even if $\beta \gg 1$, the condition (A.4) is fulfilled by taking α sufficiently large, and hence the inequality (A.3) holds for such a pair of α and β by Proposition A.1.

Remark A.1. As for the above example, the left-hand side of (A.2) is equal to

$$\frac{\beta^2(8\beta - 9)}{216\alpha^2} \wedge 0,$$

which gives a sharper condition on α and β for (A.3) to hold.

We proceed to the proof of Proposition A.1. In what follows we denote

$$U_V(x) = \frac{1}{2} \sigma^2 V'(x)^2 + x V'(x) - V(x), \quad x \in \mathbb{R}.$$

We also denote by F_X the distribution function of the random variable X :

$$F_X(x) = \frac{1}{Z} \int_{-\infty}^x e^{-V(y)} \nu(dy), \quad x \in \mathbb{R}.$$

We define

$$g := F_X^{-1} \circ \Phi, \quad (\text{A.5})$$

where F_X^{-1} is the inverse function of F_X and Φ is the standard normal cumulative distribution function:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{y^2}{2}\right) dy, \quad x \in \mathbb{R}.$$

Lemma A.1. *Suppose that for all $x \in \mathbb{R}$,*

$$U_V(x) \geq \log Z. \quad (\text{A.6})$$

Then the inequality (A.3) holds for any convex function ψ on \mathbb{R} .

Proof. In view of the proof of [14, Theorem 1.1], it suffices to show that

$$g'(x) \leq \sigma \quad \text{for all } x \in \mathbb{R}. \quad (\text{A.7})$$

Indeed, if (A.7) has been proven, then using Bass' solution [2] to the Skorokhod embedding problem, one finds that for a given one-dimensional standard Brownian motion $B = \{B(t)\}_{t \geq 0}$, there exists a stopping time T with respect to the natural filtration of B such that

$$X - E[X] \stackrel{(d)}{=} B(T) \quad \text{and} \quad T \leq \sigma^2 \quad \text{a.s.}$$

Then the inequality (A.3) is immediate either from the optional sampling theorem applied to the submartingale $\{\psi(B(t))\}_{t \geq 0}$, or from the Itô-Tanaka formula. See [14, Subsection 2.1] for details.

We turn to the proof of (A.7). The reasoning is the same as in the proof of [14, Lemma 2.1]. Since

$$g'(x) = \frac{\Phi'(x)}{F'_X \circ F_X^{-1}(\Phi(x))}$$

by the definition (A.5) of g , the inequality (A.7) is equivalent to

$$G(\xi) := \sigma F'_X \circ F_X^{-1}(\xi) - \Phi' \circ \Phi^{-1}(\xi) \geq 0 \quad \text{for all } \xi \in (0, 1). \quad (\text{A.8})$$

First note that

$$G(0+) = \lim_{\xi \rightarrow 0+} G(\xi) = 0, \quad G(1-) = \lim_{\xi \rightarrow 1-} G(\xi) = 0 \quad (\text{A.9})$$

because $\Phi' \circ \Phi^{-1}$ satisfies $\Phi' \circ \Phi^{-1}(0+) = \Phi' \circ \Phi^{-1}(1-) = 0$ and so does $F'_X \circ F_X^{-1}$ by (A.1). We now suppose that G has a local minimum at some $\xi_0 \in (0, 1)$. Then, since

$$G'(\xi) = -\left(\frac{x}{\sigma} + \sigma V'(x)\right) \Big|_{x=F_X^{-1}(\xi)} + \Phi^{-1}(\xi), \quad \xi \in (0, 1),$$

we have

$$\Phi^{-1}(\xi_0) = \left(\frac{x}{\sigma} + \sigma V'(x) \right) \Big|_{x=F_X^{-1}(\xi_0)}.$$

Therefore by the definition of G ,

$$\begin{aligned} G(\xi_0) &= \left\{ \sigma F'_X(x) - \Phi' \left(\frac{x}{\sigma} + \sigma V'(x) \right) \right\} \Big|_{x=F_X^{-1}(\xi_0)} \\ &= \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{x^2}{2\sigma^2} - V(x) \right) \left\{ \frac{1}{Z} - \exp(-U_V(x)) \right\} \Big|_{x=F_X^{-1}(\xi_0)}, \end{aligned}$$

which is nonnegative by the assumption. Combining this with (A.9) shows (A.8) and concludes the proof. \square

Using Lemma A.1, we prove Proposition A.1

Proof of Proposition A.1. The latter assertion is immediate from the fact that

$$\begin{aligned} U_V(x) &= \frac{1}{2} \left(\sigma V'(x) + \frac{x}{\sigma} \right)^2 - \frac{x^2}{2\sigma^2} - V(x) \\ &\geq -\frac{x^2}{2\sigma^2} - V(x) \end{aligned}$$

for all $x \in \mathbb{R}$. To show the former, take an arbitrary $x_0 \in \mathbb{R} \setminus \mathcal{D}_V$, namely x_0 is such that $V''(x_0) > 0$. First we suppose that

$$V(x) > V'(x_0)(x - x_0) + V(x_0)$$

for all $x \in \mathbb{R}$ but x_0 . Then as

$$\begin{aligned} Z &= \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} \exp \left(-\frac{x^2}{2\sigma^2} - V(x) \right) dx \\ &\leq \frac{1}{\sqrt{2\pi}\sigma} \exp(x_0 V'(x_0) - V(x_0)) \int_{\mathbb{R}} \exp \left(-\frac{x^2}{2\sigma^2} - V'(x_0)x \right) dx \\ &= \exp(U_V(x_0)), \end{aligned}$$

the inequality (A.6) holds for $x = x_0$. Next we suppose that

$$V(x_1) = V'(x_0)(x_1 - x_0) + V(x_0)$$

for some $x_1 \neq x_0$, say, $x_1 > x_0$. Let $x_2 \in [x_0, x_1]$ be a maximal point of the function

$$f(x) := V(x) - V'(x_0)(x - x_0) - V(x_0), \quad x \in [x_0, x_1].$$

Then it is clear that $f'(x_2) = 0$ and $f''(x_2) \leq 0$; indeed, if either of them were not the case, it would contradict the fact that x_2 is the maximal point. Therefore we have

$$V'(x_0) = V'(x_2) \tag{A.10}$$

and

$$x_2 \in \mathcal{D}_V. \quad (\text{A.11})$$

Moreover, since

$$f(x_2) = V(x_2) - V'(x_0)(x_2 - x_0) - V(x_0) \geq f(x_0) = 0,$$

it also holds that by (A.10),

$$x_0 V'(x_0) - V(x_0) \geq x_2 V'(x_2) - V(x_2).$$

Combining this inequality with (A.10), we have

$$\begin{aligned} U_V(x_0) &\geq U_V(x_2) \\ &\geq \log Z, \end{aligned}$$

where the second line is due to (A.11) and the assumption (A.2). Consequently, (A.6) holds for all $x \in \mathbb{R} \setminus \mathcal{D}_V$, and hence for all $x \in \mathbb{R}$ by (A.2). Now the assertion of the proposition follows from Lemma A.1. \square

Remark A.2. If one wants to apply the above discussion to the multidimensional case in order to extend the Brascamp-Lieb inequality (3.2) to nonconvex potentials, it would be required to draw a condition on V under which the function \tilde{V} defined by

$$\tilde{V}(x) = -\log E \left[e^{-V(Y)} \mid \alpha \cdot Y = x \right], \quad x \in \mathbb{R},$$

fulfills either the assumption (A.2) or (A.4) of Proposition A.1 with $\sigma^2 = \alpha \cdot \Sigma \alpha$. Our original motivation to extend the variational representation (1.1) to unbounded functionals stems from our desire to understand better the quantitative nature of \tilde{V} as well as that of the partition function Z .

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